Sharpness of Zapolsky inequality for quasi-states and Poisson brackets

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Abstract

Zapolsky inequality gives a lower bound for the L_1 norm of the Poisson bracket of a pair of C^1 functions on the two-dimensional sphere by means of quasi-states. Here we show that this lower bound is sharp.

1 Introduction and main results

1.1 Quasi-states and quasi-measures

Denote by $C(S^2)$ the Banach algebra of real continuous functions on S^2 taken with the supremum norm.

For $F \in C(S^2)$, write $C(F) = \{\varphi \circ F | \varphi \in C(Im(F))\}$. That is, $C(F)$ is the closed sub-algebra generated by F and the constant function 1.

Definition 1. A *quasi-state* on S^2 is a functional $\zeta : C(S^2) \to \mathbb{R}$ satisfying:

1. $\zeta(F) \geq 0$ for $F \geq 0$.
2. $\forall F \in C(S^2)$, ζ is linear on $C(F)$.
3. $\zeta(1) = 1$.

Denote by $\mathcal{Q}(S^2)$ the collection of quasi-states on S^2.

Remark 1. It was proven in [1] that for a quasi-state ζ and a pair $F, G \in C(S^2)$ we have:

$$F \leq G \Rightarrow \zeta(F) \leq \zeta(G) .$$

A quasi-state ζ is *simple* if for every $F \in C(S^2)$, ζ is multiplicative on $C(F)$. A quasi-state ζ is *representable* if it is the limit of a net of convex combinations of simple quasi-states. That is, ζ is an element of the closed convex hull of the subset of simple quasi-states.

Denote by $\mathcal{C}$ and $\mathcal{O}$ the collections of closed and open subsets of S^2 respectively. Write $\mathcal{A} = \mathcal{C} \cup \mathcal{O}$.

Definition 2. A *quasi-measure* τ on S^2 is a function $\tau : \mathcal{A} \to [0, 1]$ satisfying:

1. $\tau(S^2) = 1$.
2. For $B_1, B_2 \in \mathcal{A}$ with $B_1 \subset B_2$, $\tau(B_1) \leq \tau(B_2)$.
3. If $\{A_k\}_{k=1}^n \subset \mathcal{A}$ is a finite collection of pairwise disjoint subsets whose union is in $\mathcal{A}$, then $\tau(\bigcup_{k=1}^n A_k) = \sum_{k=1}^n \tau(A_k)$.
4. For $U \in \mathcal{O}$, $\tau(U) = \sup \{\tau(K) \ : \ K \in \mathcal{C} \text{ and } K \subset U\}$.

Denote by $\mathcal{M}(S^2)$ the collection of quasi-measures on S^2. A quasi-measure is *simple* if it only takes values of 0 and 1.

It was proven in [1] that there exists a bijection between $\mathcal{Q}(S^2)$ and $\mathcal{M}(S^2)$. For a quasi-state ζ, the corresponding quasi-measure is:

$$\tau(A) = \begin{cases} A \in \mathcal{C}, & \inf \{\zeta(F) \ : \ F \in C(S^2) \text{ and } F \geq 1_A\} \\ A \in \mathcal{O}, & 1 - \tau(S^2 \setminus A) \end{cases} ,$$

here 1_A is the indicator function on the set A. The corresponding quasi-state to a quasi-measure τ is defined as follows:

$$\zeta(F) = \int_{S^2} F d\tau = \max_{S^2} F - \int_{\min_{S^2} F}^{\max_{S^2} F} b_F(x) dx ,$$

with $b_F(x) = \tau(\{F < x\})$. It was proven in [2] that this bijection matches simple quasi-states with simple quasi-measures. For further details about quasi-states and quasi-measures refer to [1] and for details on simple quasi-states and quasi-measures refer to [2].

Throughout this paper we will be interested in the extent of non-linearity of a quasi-state. To measure this we will use the following notation:

Definition 3. Let ζ be a quasi-state and take $F, G \in C(S^2)$. The extent of non-linearity of ζ can be measured by:

$$\Pi(F,G) := |\zeta(F+G) - \zeta(F) - \zeta(G)| \ .$$

Example 1. One example of a simple quasi-state is Aarnes' 3-point quasi-state.

Definition 4. A subset $S \subseteq S^2$ is called *solid* if it is connected and its complement $S^c = S^2 \setminus S$ is also connected. Denote by $\mathcal{C}_s$ the set of all closed and solid subsets of S^2 and by $\mathcal{O}_s$ the set of all open and solid subsets of S^2. Write $\mathcal{A}_s = \mathcal{C}_s \cup \mathcal{O}_s$.

Take $p_1, p_2, p_3 \in S^2$ to be three distinct points on the sphere. Define $\tau : \mathcal{C}_s \to \{0,1\}$ by:

$$\tau(C) = \begin{cases} 0, & \#\{C \cap \{p_1,p_2,p_3\}\} \leq 1 \\ 1, & \#\{C \cap \{p_1,p_2,p_3\}\} \geq 2 \end{cases} \ .$$

As proved in [3], τ can be extended to a quasi-measure on S^2. It is further shown in that article that this extension is in-fact a simple quasi-measure. The simple quasi-state corresponding to the extended quasi-measure is called Aarnes' 3-point quasi-state. We refer the reader to [3] for the full definition of the extended quasi-measure τ. For our purpose it suffices to note that on $\mathcal{A}_s$, τ satisfies:

$$\tau(S) = \begin{cases} 0, & \#\{S \cap \{p_1,p_2,p_3\}\} \leq 1 \\ 1, & \#\{S \cap \{p_1,p_2,p_3\}\} \geq 2 \end{cases} \ .$$

Example 2. Another example of a simple quasi-state is the median of a Morse function. Let Ω be an area form on S^2. The *median* of a Morse function F is the unique connected component of a level set of F, m_F, for which every connected component of $S^2 \setminus m_F$ has area $\leq \frac{1}{2} \cdot \int_{S^2} \Omega$. Define ζ on the set of Morse functions as $\zeta(F) = F(m_F)$. As explained in [5], ζ can be extended to $C(S^2)$ and is in-fact a quasi-state. For further explanation of the concept of the median and the construction of ζ we refer the reader to [5]. It can be easily verified that the quasi-measure corresponding to ζ is the extension of $\tau : \mathcal{C}_s \to \{0,1\}$ defined as:

$$\tau(C) = \begin{cases} 0, & \int_C \Omega < \frac{1}{2} \cdot \int_{S^2} \Omega \\ 1, & \int_C \Omega \geq \frac{1}{2} \cdot \int_{S^2} \Omega \end{cases}$$

to a quasi-measure on S^2 as in [3]. In-fact, as explained in [3], this extension is a simple quasi-measure, and hence ζ is a simple quasi-state.

1.2 Poisson bracket

Let ω be an area form on S^2. Given a hamiltonian $F : S^2 \to \mathbb{R}$, we define the hamiltonian vector field $IdF : S^2 \to TS^2$ by the formula:

$$dF(x)(\eta) = \omega(\eta, IdF(x)) \ , \forall x \in S^2, \eta \in T_x S^2 \ .$$

The hamiltonian flow with hamiltonian function F is the one-parameter group of diffeomorphisms $\{{g^t}_F\}$ satisfying:

$$\left.\frac{d}{dt}\right|_{t=0} {g^t}_F x = IdF(x) \ .$$

If F, G are two hamiltonian functions on S^2, then their *Poisson bracket* is defined as:

$$\{F,G\}(x) = \left.\frac{d}{dt}\right|_{t=0} F({g^t}_G(x)) \ .$$

The Poisson bracket also satisfies the following formula:

$$\{F,G\} = dF(IdG) = -\omega(IdF, IdG) \ .$$

For further reading on Poisson bracket we refer the reader to [4].

Remark 2. In this paper we are interested in the L_1-norm of the Poisson bracket. Note that on S^2 we have:

$$dF \wedge dG = -\{F, G\} \cdot \omega \ ,$$

therefore:

$$\|\{F, G\}\|_{L_1} = \int_{S^2} |\{F, G\}| \, \omega = \int_{S^2} |dF \wedge dG| \ .$$

1.3 Zapolsky's inequality

Zapolsky's inequality ([9], theorem 1.4) relates the extent of non-linearity of a quasi-state to the L_1 norm of the Poisson bracket. Let ζ be a representable quasi-state on S^2, then by Zapolsky's inequality for every $F, G \in C^1(S^2)$ we have:

$$\Pi(F, G)^2 \leq \| \{F, G\} \|_{L_1} \ .$$

Note that this result can also be written as:

$$\sup_{F,G \in C^1(S^2)} \frac{\Pi(F, G)^2}{\| \{F, G\} \|_{L_1}} \leq 1 \ .$$

Our goal in this paper is to show that for some quasi-states Zapolsky's inequality is sharp. That is, we will show that there exist quasi-states for which:

$$\sup_{F,G \in C^1(S^2)} \frac{\Pi(F, G)^2}{\| \{F, G\} \|_{L_1}} = 1 \ .$$

1.4 Main Results

Theorem 1. *Let ζ be Aarnes' 3-point quasi-state, then:*

$$\max_{F,G \in C^\infty(S^2)} \frac{\Pi(F, G)^2}{\| \{F, G\} \|_{L_1}} = 1 \ .$$

Theorem 2. *Let ω be a normalized area form on S^2, that is $\int_{S^2} \omega = 1$, and ζ the corresponding median quasi-state. Then we have:*

$$\sup_{F,G \in C^\infty(S^2)} \frac{\Pi(F,G)^2}{\| \{F,G\} \|_{L_1}} = 1 \ .$$

2 Proofs

2.1 Proof of theorem 1

Prior to proving theorem 1 we shall pay attention to the fact that any result we can prove for a certain 3-point quasi-state is true for all such quasi-states.

Remark 3. Let $\{p_1, p_2, p_3\}$ and $\{q_1, q_2, q_3\}$ be two sets of three distinct points on the sphere S^2, and take ζ_1 and ζ_2 to be the two corresponding Aarnes' 3-point quasi-states and Π_1 and Π_2 the corresponding measurements of their non-linearity. By a corollary to the isotopy lemma (see [6], 3.6) there exists a diffeomorphism $h : S^2 \to S^2$ satisfying:

$$h(p_i) = q_i \ , \ 1 \leq i \leq 3 \ .$$

Since h is a diffeomorphism, both h and h^{-1} take solid subsets of the sphere to solid subsets, thus $\zeta_2(F \circ h) = \zeta_1(F)$ for every function $F \in C(S^2)$. Which yields:

$$\Pi_1(F,G) = \Pi_2(F \circ h, G \circ h) \ .$$

Also, we have:

$$\| \{F \circ h, G \circ h\} \|_{L_1} = \int_{S^2} |d(F \circ h) \wedge d(G \circ h)| = \int_{S^2} |h^* (dF \wedge dG)| = \int_{h(S^2)} |dF \wedge dG| = \int_{S^2} |dF \wedge dG| = \| \{F,G\} \|_{L_1} \ .$$

Thus:

$$\frac{\Pi_2(F \circ h, G \circ h)^2}{\| \{F \circ h, G \circ h\} \|_{L_1}} = \frac{\Pi_1(F,G)^2}{\| \{F,G\} \|_{L_1}} \ .$$

Based on this result we can prove the following theorem for a certain 3-point quasi-state and conclude that it is true for all such quasi-states.

Proof of theorem 1

Proof. Define $S^2 = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. In spherical coordinates we have:

$$S^2 = \left\{(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta) \ : \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \text{ and } 0 \le \phi \le 2\pi\right\} \ .$$

Consider the following points on S^2:

$$\begin{array}{rcl} p_1 & = & (1,0,0) \\ p_2 & = & (0,1,0) \\ p_3 & = & (0,0,1) \end{array} \ .$$

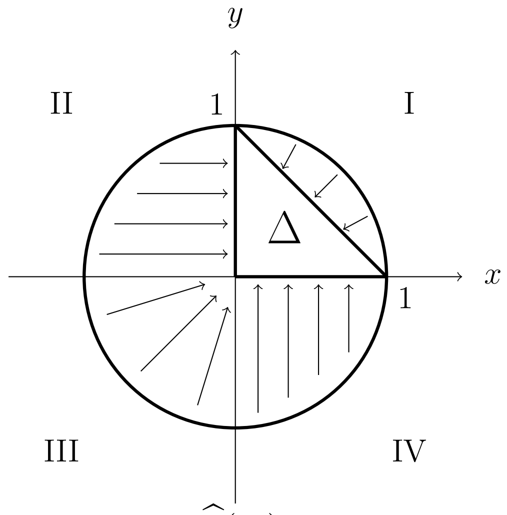
Let ζ and τ be Aarnes' 3-point quasi-state and quasi-measure corresponding to these points.

Denote:

$$\begin{array}{l} \mathcal{D} = \{(x,y) : x^2 + y^2 \le 1\} \\ \Delta = \{(u,v) \in \mathbb{R}^2 : u, v > 0 \text{ and } u + v < 1\} \end{array} \ .$$

We build a continuous function $\widehat{\psi} : \mathcal{D} \to cl(\Delta)$ (see the figure below) satisfying :

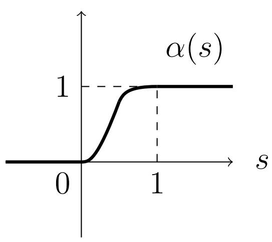
- $\widehat{\psi}$ maps the first quarter homeomorphically to Δ along the radii.
- $\widehat{\psi}$ maps the second quarter to the segment $\{0\} \times [0,1]$ of the y-axis.
- $\widehat{\psi}$ maps the third quarter to the origin $(0,0)$.
- $\widehat{\psi}$ maps the fourth quarter to the segment $[0,1] \times \{0\}$ of the x-axis.



$\widehat{\psi}(\mathcal{D})$

We now build a smooth function $\psi : \mathcal{D} \to cl(\Delta)$ by smoothening $\widehat{\psi}$. For the precise definition of ψ we will need an auxiliary smooth function $\alpha : \mathbb{R} \to [0, 1]$ satisfying:

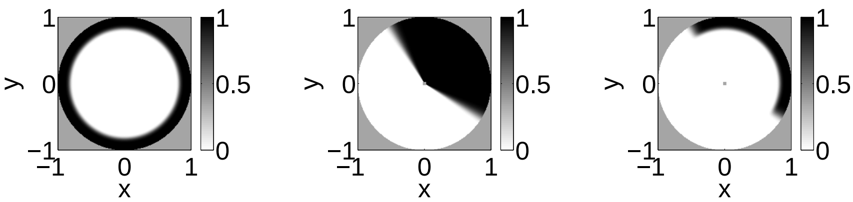
$$
\begin{aligned}
&\alpha(s) = 0 \ , \ \forall s \leq 0 \\
&\alpha(s) = 1 \ , \ \forall s \geq 1 \\
&\alpha'(s) > 0 \ , \ \forall s \in (0, 1).
\end{aligned}
$$



Then we can define:

$$
\rho(x, y) = \alpha\left(2x^2 + 2y^2 - 1\right) \cdot \alpha\left(\frac{x + y}{\sqrt{x^2 + y^2}}\right) \ , \ \forall (x, y) \in \mathcal{D} \setminus \{(0, 0)\} \ ,
$$

the images below illustrate the behaviour of this function.



$\alpha(2x^2 + 2y^2 - 1)$ $\alpha\left(\frac{x+y}{\sqrt{x^2+y^2}}\right)$ $\rho(x, y)$

And we take $\psi : \mathcal{D} \to cl(\Delta)$ to be $\psi(p) = (f(p), g(p))$, whereas:

$$
\begin{aligned}
f(x,y) &= \begin{cases} x \le 0 &, \quad 0 \\ 0 < x &, \quad \rho(x,y) \cdot \dfrac{\alpha\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{\alpha\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \alpha\left(\frac{y}{\sqrt{x^2+y^2}}\right)} \end{cases} \\
g(x,y) &= \begin{cases} y \le 0 &, \quad 0 \\ 0 < y &, \quad \rho(x,y) \cdot \dfrac{\alpha\left(\frac{y}{\sqrt{x^2+y^2}}\right)}{\alpha\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \alpha\left(\frac{y}{\sqrt{x^2+y^2}}\right)} \end{cases}
\end{aligned}
.
$$

Lemma 1. *f, g are smooth functions.*

Proof. The proofs of smoothness for f and g are very similar, therefore we will give the proof only for f. To show that f is smooth, we need to show that it is smooth on every point of its domain. Take a point $(x_0, y_0) \in \mathcal{D}$, and consider the following cases:

- If $x_0 < 0$, then f is identically zero in a neighbourhood of x_0, and hence smooth.
- If $x_0 > 0$, then $x > 0$ in a neighbourhood of x_0, hence $\sqrt{x^2 + y^2} > 0$ and f is a multiplication of smooth functions divided by smooth $\alpha\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \alpha\left(\frac{y}{\sqrt{x^2+y^2}}\right)$. But, since $x > 0$, $\alpha\left(\frac{x}{\sqrt{x^2+y^2}}\right) > 0$, therefore the denominator $\alpha\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \alpha\left(\frac{y}{\sqrt{x^2+y^2}}\right) > 0$, and f is smooth.
- If $x_0 = 0$ and $y_0 < 0$, we can find a neighbourhood U of (x_0, y_0) on which $y < 0$ and $x + y \le 0$. But then in this neighbourhood we have $x^2 + y^2 > 0$ and $\frac{x+y}{\sqrt{x^2+y^2}} \le 0$, hence $\alpha\left(\frac{x+y}{\sqrt{x^2+y^2}}\right) = 0 \Rightarrow \rho(x,y) = 0$,

which yields:

$$f(x,y)|_U = \begin{cases} x>0 \quad , \quad 0 \cdot \dfrac{\alpha\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{\alpha\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\alpha\left(\frac{y}{\sqrt{x^2+y^2}}\right)} = 0 \\ \\ x \leq 0 \quad , \quad 0 \end{cases} .$$

Thus f is identically zero in this neighbourhood, and hence smooth.

- If $x_0 = 0$ and $y_0 > 0$, we can find a neighbourhood of y_0 such that $y > 0$, thus $\alpha\left(\frac{y}{\sqrt{x^2+y^2}}\right) > 0$. In this neighbourhood the denominator, $\alpha\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\alpha\left(\frac{y}{\sqrt{x^2+y^2}}\right) > 0$, thus f will be the multiplication of smooth functions for $x > 0$ and zero for $x \leq 0$. From α's smoothness we have $\lim_{s\to 0} \alpha^{(m)}(s) = 0$ for every derivative $m \in \mathbb{N}$. If $x > 0$, every derivative of f will be a finite sum of products, each of which has a multiplicand of the form $\alpha^{(m)}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$ for some $m \in \mathbb{N}$, therefore:

$$\lim_{(x,y)\to(0,y_0)} f^{(n)}(x,y) = 0 \ , \ \forall n \in \mathbb{N} \ ,$$

  and f is smooth.

- Finally, if $x_0 = 0$ and $y_0 = 0$, we can find a neighbourhood of (x_0, y_0) on which we have $x^2 + y^2 < \frac{1}{2}$. But then: $\alpha(2x^2 + 2y^2 - 1) = 0 \Rightarrow \rho(x,y) = 0$, and hence f is identically zero in this neighbourhood, thus smooth.

We have shown that f is smooth on every point of $\mathcal{D}$, thus f is a smooth function. In a similar manner it can be shown that g is also smooth. $\square$

Denote:

$$A = \left\{(x,y) \in \mathcal{D} : x, y > 0 \text{ and } \frac{1}{2} < x^2 + y^2 < 1\right\} .$$

Lemma 2. *The restriction* $\psi|_A$ *is one-to-one and onto* Δ*. Also,* $\psi(\mathcal{D} \setminus A) \subset \partial\Delta$*.*

Proof. On A we have $x, y > 0$, thus $x + y > \sqrt{x^2 + y^2}$ and $\alpha\left(\frac{x+y}{\sqrt{x^2+y^2}}\right) = 1$. Hence:

$$(f+g)|_A = \rho|_A = \alpha(2x^2 + 2y^2 - 1) \ .$$

Similarly:

$$\left(\frac{f}{g}\right)\Big|_A = \frac{\alpha\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{\alpha\left(\frac{y}{\sqrt{x^2+y^2}}\right)} \ .$$

In spherical coordinates we have:

$$A = \left\{(\cos\theta\cos\phi, \cos\theta\sin\phi) : 0 < \phi < \frac{\pi}{2} \text{ and } 0 < \theta < \frac{\pi}{4}\right\} \ .$$

Therefore:

$$(f+g)|_A = \alpha(2cos^2\theta - 1) \text{ and } \left(\frac{f}{g}\right)\Big|_A = \frac{\alpha(\cos\phi)}{\alpha(\sin\phi)} \ .$$

Note that α is a bijection of $(0,1)$ to $(0,1)$ and $(2\cos^2\theta - 1)$ is a bijection of $\left(0, \frac{\pi}{4}\right)$ to $(0,1)$, hence $(f+g)\,|_A = \alpha(\cos^2\theta - 1)$ is a bijection of $\left(0, \frac{\pi}{4}\right)$ to $(0,1)$. Also:

$$\frac{d\left(\frac{f}{g}\right)\Big|_A}{d\phi} = \frac{d\left(\frac{\alpha(\cos\phi)}{\alpha(\sin\phi)}\right)}{d\phi} = - \frac{\alpha'(\cos\phi)\cdot\alpha(\sin\phi)\cdot sin\phi + \alpha(\cos\phi)\cdot\alpha'(\sin\phi)\cdot\cos\phi}{\alpha^2(\sin\phi)}$$

Recall that $\alpha(s), \alpha'(s) > 0$ for $s \in (0,1)$ and that on A we have $0 < \cos\phi, \sin\phi < 1$, therefore: $\dfrac{d\left(\frac{f}{g}\right)\Big|_A}{d\phi} < 0$, and $\left(\frac{f}{g}\right)\Big|_A$ is a bijection of $\left(0, \frac{\pi}{2}\right)$ to $(0, \infty)$.

We have shown that $\left(f+g, \frac{f}{g}\right)\Big|_A$ is a bijection of A to $(0,1)\times(0,\infty)$. Since $\left(u+v, \frac{u}{v}\right)$ is a bijection of Δ to $(0,1)\times(0,\infty)$, $\psi|_A$ is a bijection of A to Δ.

We still have to show that $\psi(\mathcal{D}\setminus A)\subset\partial\Delta$. Note that a point $(x,y)\in\mathcal{D}\setminus A$ satisfies at-least one of these four conditions:

- $x\le 0$
  In this case we have $f(x,y)=0$ and $\psi(x,y)=(0,g(x,y))\in\partial\Delta$.
- $y\le 0$
  Similarly $g(x,y)=0$ and $\psi(x,y)=(f(x,y),0)\in\partial\Delta$.
- $x^2+y^2\le\frac{1}{2}$
  Here $\rho(x,y)=\alpha(2x^2+2y^2-1)\cdot\alpha(\frac{x+y}{\sqrt{x^2+y^2}})=0$ and $\psi(x,y)=(0,0)\in\partial\Delta$.
- $x,y>0$ and $x^2+y^2=1$
  Here $\rho(x,y)=\alpha(2x^2+2y^2-1)\cdot\alpha\left(\frac{x+y}{\sqrt{x^2+y^2}}\right)=1$. hence $f(x,y)+g(x,y)=1$ and $\psi(x,y)=(f(x,y),g(x,y))\in\partial\Delta$.

Thus we have shown that $\psi(\mathcal{D}\setminus A)\subset\partial\Delta$. $\square$

Let $P:S^2\to\mathbb{R}^2$ be the projection of the sphere to the xy-plane. Define: $F,G:S^2\to\mathbb{R}$ by $F=f\circ P$ and $G=g\circ P$. Our goal is to show that:

$$\Pi^2(F,G)=\|\{F,G\}\|_{L_1}\ .$$

We will begin by proving the following lemma:

Lemma 3.

$$\Pi(F,G)=1\ .$$

Proof. Note:

$$\begin{array}{rcl}(F,G)(p_1)&=&(1,0)\\(F,G)(p_2)&=&(0,1)\\(F,G)(p_3)&=&(0,0)\end{array}\ .$$

Since $p_2, p_3 \in \{(x,y,z) \in S^2 : x \leq 0\}$, and since the half-sphere is a solid subset of the sphere we have $\tau(\{(x,y,z) \in S^2 : x \leq 0\}) = 1$. Also:

$$\Big\{(x,y,z) \in S^2 : x \leq 0\Big\} \subset F^{-1}(0) \subset \{F < t\} \ , \ \forall t > 0 \ .$$

Therefore:

$$b_F(t) = \tau(\{F < t\}) = 1 \ , \ \forall t > 0 \ .$$

In the same way we have $p_1, p_3 \in \{(x,y,z) \in S^2 : y \leq 0\}$, and as this half-sphere is also a solid subset, we get once more $\tau(\{(x,y,z) \in S^2 : y \leq 0\}) = 1$. As before:

$$\Big\{(x,y,z) \in S^2 : y \leq 0\Big\} \subset G^{-1}(0) \subset \{G < t\} \ , \ \forall t > 0 \ .$$

Thus:

$$b_G(t) = \tau(\{G < t\}) = 1 \ , \ \forall t > 0 \ .$$

Last it should be noted that the arc:

$$\Big\{(x,y,0) \in S^2 : x, y \geq 0 \text{ and } x^2 + y^2 = 1\Big\}$$

is also a solid subset of the sphere, and that:

$$p_1, p_2 \in \Big\{(x,y,0) \in S^2 : x, y \geq 0 \text{ and } x^2 + y^2 = 1\Big\} \ .$$

Therefore:

$$\tau(\Big\{(x,y,0) \in S^2 : x, y \geq 0 \text{ and } x^2 + y^2 = 1\Big\}) = 1 \ .$$

Since:

$$\Big\{(x,y,0) \in S^2 : x, y \geq 0 \text{ and } x^2 + y^2 = 1\Big\} \subset (F+G)^{-1}(1) \ ,$$

we have:

$$\tau((F+G)^{-1}(1)) = 1 \ .$$

Therefore, the quasi-measure of its complement $\{F+G<1\}$ is 0. For every $0 \leq t \leq 1$, $\{F+G<t\}$ is a subset of $\{F+G<1\}$, thus:

$$b_{F+G}(t) = \tau(\{F+G<t\}) = 0, \forall t \leq 1 \ .$$

Hence:

$$\begin{aligned}
\zeta(F) &= 1 - \int_0^1 b_F(t)dt = 1 - \int_0^1 1dt = 0 \\
\zeta(G) &= 1 - \int_0^1 b_G(t)dt = 1 - \int_0^1 1dt = 0 \\
\zeta(F+G) &= 1 - \int_0^1 b_{F+G}(t)dt = 1 - \int_0^1 0dt = 1 \ .
\end{aligned}$$

And we get that:

$$\Pi(F,G) = |\zeta(F+G) - \zeta(F) - \zeta(G)| = |1-0-0| = 1 \ .$$

$\square$

We now have to compute $\| \{F,G\} \|_{L_1}$. Recall that:

$$\| \{F,G\} \|_{L_1} = \int_{S^2} |dF \wedge dG| = \int_{S^2} |(\psi \circ P)^*(dx \wedge dy)| \ .$$

From lemma 1, $\psi \circ P$ is a smooth function, then, as a corollary to the change of variables formula for a many-to-one function (see [8], theorem F.1) we have:

$$\int_{S^2} |(\psi \circ P)^*(dx \wedge dy)| = \int_{\psi \circ P(S^2)} n(x,y) \cdot dx \wedge dy \ ,$$

with:

$$n(x,y) = card((\psi \circ P)^{-1}(x,y)) \ .$$

Also, by lemma 2, we know that $\psi \circ P$ covers Δ exactly twice (since P projects the sphere twice onto A), hence $n(x,y) = 2$ for $(x,y) \in \Delta$. Thus:

$$\int_{S^2} |(\psi \circ P)^*(dx \wedge dy)| = \int_{cl(\Delta)} n(x,y) dx \wedge dy = \int_{\Delta} 2dx \wedge dy = 1 \ .$$

Thus we have shown that for Aarnes' 3-point quasi-state corresponding to these specific three points p_1, p_2, p_3 we have:

$$\frac{\Pi(F,G)^2}{\| \{F,G\} \|_{L_1}} = \frac{1^2}{1} = 1 \ .$$

Remark 3 concludes this proof for any 3-point quasi-state. $\square$

2.2 Proof of theorem 2

In the proof of theorem 2 we will use the fact that diffeomorphisms preserve the relation between the extent of non-linearity of median quasi-states and the L_1-norm of the Poisson bracket.

Remark 4. Let $h : M_1 \to M_2$ be a diffeomorphism of surfaces. If ω is an area form on M_2 then $h^*\omega$ is an area form on M_1. Take ζ_1 and ζ_2 to be the median quasi-states corresponding to $h^*\omega$ and ω. Recall that m_F, the median of a function $F \in C^1(M_2)$, is the unique connected component of the level set $F^{-1}(\zeta_2(F)) \subset M_2$ satisfying $\int_B \omega \leq \frac{1}{2}\int_{M_2} \omega$ for each connected component B of $M_2 \backslash m_F$. Since h, h^{-1} are continuous functions, they take connected sets to connected sets, therefore $h^{-1}(m_F)$ is a connected component of the level set $(F \circ h)^{-1}(\zeta_2(F)) \subset M_1$. If A is a connected component of $M_1 \setminus h^{-1}(m_F)$, then $h(A)$ must be a connected component of $M_2 \setminus (m_F)$. Therefore:

$$\int_A h^*\omega = \int_{h(A)} \omega \leq \frac{1}{2} \int_{M_2} \omega = \frac{1}{2} \int_{M_1} h^*\omega \ .$$

Thus $h^{-1}(m_F)$ must be the median of the function $F \circ h$, which yields:

$$\zeta_1(F \circ h) = \zeta_2(F) \ .$$

Therefore if Π_1 and Π_2 are the extents of non-linearity of the quasi-states ζ_1 and ζ_2, we get:

$$\Pi_1(F \circ h, G \circ h) = \Pi_2(F, G) \ .$$

Also, we have:

$$\| \{F \circ h, G \circ h\} \|_{L_1} = \int_{M_1} |d(F \circ h) \wedge d(G \circ h)| = \int_{M_1} |h^* (dF \wedge dG)| =$$
$$\int_{h(M_1)} |dF \wedge dG| = \int_{M_2} |dF \wedge dG| = \| \{F \circ G\} \|_{L_1} \ .$$

Thus:

$$\frac{\Pi_1(F \circ h, G \circ h)^2}{\| \{F \circ h, G \circ h\} \|_{L_1}} = \frac{\Pi_2(F, G)^2}{\| \{F, G\} \|_{L_1}} \ ,$$

and:

$$\sup_{F,G \in C^\infty(M_1)} \frac{\Pi_1(F, G)^2}{\| \{F, G\} \|_{L_1}} = \sup_{F,G \in C^\infty(M_2)} \frac{\Pi_2(F, G)^2}{\| \{F, G\} \|_{L_1}} \ .$$

Proof of theorem 2

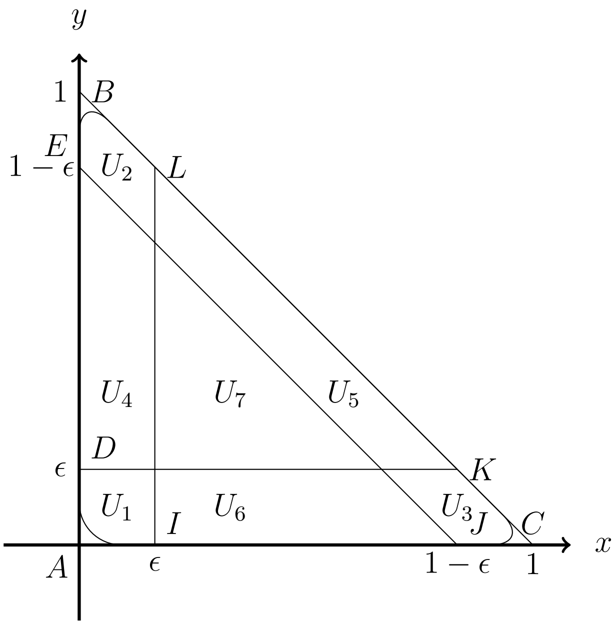
Proof. Consider the triangle ABC with vertices:

$$\begin{cases} A &= (0,0) \\ B &= (0,1) \\ C &= (1,0) \end{cases}$$

in the xy-plane. For $\frac{1}{4} > \epsilon > 0$ draw the segments DK, EJ, IL with:

$$\begin{cases} D = (0,\epsilon) & K = (1-\epsilon,\epsilon) \\ E = (0,1-\epsilon) & J = (1-\epsilon,0) \\ I = (\epsilon,0) & L = (\epsilon,1-\epsilon) \end{cases} .$$

Let U be the triangle $\triangle ABC$ after smoothing its corners by curves that do not intersect the segments DK, EJ and IL. Then the segments DK, EJ and IL divide U into seven parts, $U_1, U_2, \ldots, U_7$.



Note that $U_7 \subset U \subset \triangle ABC$, and hence:

$$\frac{(1-3\epsilon)^2}{2} = Area(U_7) < Area(U) < Area(\triangle ABC) = \frac{1}{2} . \quad (1)$$

Let $u : U \to [0, \infty)$ be a function satisfying $u^{-1}(0) = \partial U$ with 0 a regular value of u. And take S to be the surface in $\mathbb{R}^3$ defined as $S := \{z^2 = u(x, y)\}$.
Consider the following functions:

- $P : S \to \mathbb{R}^2$ defined as $P(x, y, z) = (x, y)$ is the projection of S to the plane. Note that $S \setminus P^{-1}(\partial U)$ has two connected components,

$$\left\{(x, y, \pm\sqrt{u(x, y)}) : (x, y) \in int(U)\right\} ,$$

  both of which are projected diffeomorphically to $int(U)$ by P.

- $F : S \to \mathbb{R}$ defined as $F(x, y, z) = x$.

- $G : S \to \mathbb{R}$ defined as $G(x, y, z) = y$.

Then by (1) we get:

$$\| \{F, G\} \|_{L_1} = \int_S |dF \wedge dG| = \int_S |dx \wedge dy| = 2 \cdot Area(U) \in ((1 - 3\epsilon)^2, 1) .$$

Let σ be an area form on S such that:

$$\int_{P^{-1}(U_1)} \sigma = \int_{P^{-1}(U_2)} \sigma = \int_{P^{-1}(U_3)} \sigma = \frac{2}{10}$$

and

$$\int_{P^{-1}(U_4)} \sigma = \int_{P^{-1}(U_5)} \sigma = \int_{P^{-1}(U_6)} \sigma = \int_{P^{-1}(U_7)} \sigma = \frac{1}{10} .$$

Note that σ is a normalized area form on S, and that each of the curves $P^{-1}(IL), P^{-1}(DK)$ and $P^{-1}(EJ)$ divides S into two disks, one of area:

$$\frac{2}{10} + \frac{1}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}$$

and the second of area:

$$\frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2} .$$

Then, if ζ is the median quasi-state corresponding to σ, we get:

$$\begin{cases} \zeta(F) & = & F(IL) & = & \epsilon \\ \zeta(G) & = & G(DK) & = & \epsilon \\ \zeta(F+G) & = & (F+G)(EJ) & = & 1-\epsilon \end{cases} .$$

Therefore:

$$\frac{\Pi(F,G)^2}{\| \{F,G\} \|_{L_1}} \geq \frac{|1-\epsilon-\epsilon-\epsilon|^2}{1} \xrightarrow[\epsilon\to 0]{} 1 ,$$

and hence we have:

$$\sup_{F,G\in C^\infty(S)} \frac{\Pi(F,G)^2}{\| \{F,G\} \|_{L_1}} = 1 .$$

Note that U is diffeomorphic to a closed disk, hence S is diffeomorphic to the sphere, and there exists a diffeomorphism $h_1 : S^2 \to S$. Recall that σ is a normalized area form on S, hence $\sigma_1 = {h_1}^*\sigma$ is a normalized area form on S^2. Let Π_1 be the extent of non-linearity of the median quasi-state corresponding to σ_1, then by remark 4 we have:

$$\sup_{F,G\in C^\infty(S^2)} \frac{\Pi_1(F,G)^2}{\| \{F,G\} \|_{L_1}} = \sup_{F,G\in C^\infty(S)} \frac{\Pi(F,G)^2}{\| \{F,G\} \|_{L_1}} = 1 .$$

If σ_2 is another normalized area form on S^2, then by Moser's theorem (see [7], 13.2) there exists a diffeomorphism $h_2 : S^2 \to S^2$, such that $\sigma_2 = {h_2}^*\sigma_1$. If Π_2 is the extent of non-linearity of the median quasi-state corresponding to σ_2, then by using remark 4 again, we will get:

$$\sup_{F,G\in C^\infty(S^2)} \frac{\Pi_2(F,G)^2}{\| \{F,G\} \|_{L_1}} = \sup_{F,G\in C^\infty(S^2)} \frac{\Pi_1(F,G)^2}{\| \{F,G\} \|_{L_1}} = 1 .$$

Thus, for every normalized area form ω on S^2, if Π is the extent of non-linearity of its corresponding median quasi-state, we have:

$$\sup_{F,G\in C^\infty(S^2)} \frac{\Pi(F,G)^2}{\| \{F,G\} \|_{L_1}} = 1 .$$

$\square$

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